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# Thermally induced dynamic instability of laminated composite conical shells

Chih-Ping Wu *, Shih-Jung Chiu

*Department of Civil Engineering, National Cheng Kung University, Tainan 70101, Taiwan, ROC*

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**Abstract**

Thermally induced dynamic instability of laminated composite conical shells is investigated by means of a perturbation method. The laminated composite conical shells are subjected to static and periodic thermal loads. The linear instability approach is adopted in the present study. A set of initial membrane stresses due to the elevated temperature field is assumed to exist just before the instability occurs. The formulation begins with three-dimensional equations of motion in terms of incremental stresses perturbed from the state of neutral equilibrium. After proper nondimensionalization, asymptotic expansion and successive integration, we obtain recursive sets of differential equations at various levels. The method of multiple scales is used to eliminate the secular terms and make an asymptotic expansion feasible. Using the method of differential quadrature and Bolotin's method, and imposing the orthonormality and solvability conditions on the present asymptotic formulation, we determine the boundary frequencies of dynamic instability regions for various orders in a consistent and hierarchical manner. The principal instability regions of cross-ply conical shells with simply supported–simply supported boundary conditions are studied to demonstrate the performance of the present asymptotic theory. © 2002 Elsevier Science Ltd. All rights reserved.

*Keywords:* Dynamic instability; Conical shells; Thermal loads; Perturbation; Asymptotic expansion; Differential quadrature method; Three-dimensional analysis

## 1. Introduction

Research topics related to dynamic instability of elastic systems have received substantial attention through the years (Bolotin, 1964). The subject deals with the dynamic behavior of elastic systems subjected to external static and dynamic loads. In the analyses of these problems, the boundary frequencies of dynamic instability regions for a system of generalized Mathieu-Hill equations are determined. General concepts and comprehensive investigations of various isotropic structural components can be found in the literature (Beliaev, 1924; Bolotin, 1964; Koval, 1974).

In recent decades, composite materials were increasingly used in the industrial applications. Dynamic instability of laminated composite plates and shells was therefore studied. Argento and Scott (1993a,b) and Argento (1993) analyzed the dynamic instability of layered anisotropic circular cylindrical shells under

* Corresponding author.
*E-mail address:* cpwu@mail.ncku.edu.tw (C.-P. Wu).


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periodic axial loads and combination of the periodic axial and torsional loads. The variations of instability regions with the circumferential wave number and the magnitude of external loads were investigated. Bert and Birman (1988) presented a detailed study on the dynamic instability of thick laminated cylindrical shells. The effect of transverse deformation on the principal instability region was studied. Based on Love's classical theory, Ng et al. (1998) examined the effect of the magnitude of axial loads on the instability regions. Using an extension of Donnell's shell theory to a first-order shear deformation theory (FSDT), Ng et al. (1999) estimated the effects of thickness-to-radius ratio on the instability regions. Comparisons of instability regions generated using various classical shell theories (CST) (i.e., Donnell's, Love's and Flügge's shell theories) were made by Ng and Lam (1999).

After making a literature survey, we found that most of the articles deal with dynamic instability of laminated cylindrical shells under various external periodic mechanical loads. The analysis of thermal dynamic instability has received less attention. Birman and Bert (1990) presented the dynamic instability analysis of reinforced composite cylindrical shells in thermal fields. On the basis of a FSDT, an identical study was presented by Ganapathi and Touratier (1998).

The aforementioned papers presented the two-dimensional (2D) results for the dynamic instability analysis of laminates subjected to static and periodic thermomechanical loads. A detailed study for the three-dimensional (3D) analysis of thermal dynamic instability is lacking. Hence, the present paper aims at developing a 3D formulation for the dynamic instability analysis of laminated conical shells subjected to static and periodic thermal loads by means of a perturbation method.

Asymptotic differential quadrature (DQ) solutions for the thermal dynamic instability analysis of laminated circular conical shells are presented in this paper. It is an extension to the recent studies related to asymptotic theories for free vibration (Wu and Wu, 2000) and for thermal buckling (Wu and Chiu, 2001) problems. The linear instability approach is considered in the present formulation. A geometric small perturbation parameter and a set of dimensionless field variables are defined. Through nondimensionalization, asymptotic expansion and successive integration, the asymptotic theory finally turns out recursive sets of CST governing equations for various orders. The method of multiple scales (Nayfeh, 1993) is used to eliminate the secular terms and make the asymptotic expansion feasible. Using Fourier series expansion in the circumferential coordinate, the recursive sets of governing equations can be reduced to systems of partial differential equations where the derivatives are with respect to the meridional coordinate and the time variable. According to the DQ rule, we replace the resulting governing equations and the corresponding boundary conditions for various orders as systems of generalized Mathieu-Hill equations. The solution procedure suggested by Bolotin (1964) is used to determine the boundary frequencies of dynamic instability regions at the leading-order level. Imposing of the orthonormality and solvability conditions for higher-order problems, the higher-order modifications to boundary frequencies can be uniquely determined in a hierarchical and consistent manner.

## 2. Basic three-dimensional equations

Consider a laminated composite conical shell as shown in Fig. 1. The material properties are considered to be piecewise-constant functions of the thickness coordinate. A set of the conical coordinates $(s, \theta, \zeta)$ is located on the middle surface. $R_1$ and $R_2$ are the radii of the cone at the small and large edges, respectively. $\alpha$ is semivertex angle of the cone, $2h$ denotes the shell thickness, and $L$ is the slant length of the cone.

According to the assumptions of the linear instability approach, a set of membrane state of stress exists in the shell just before instability occurs (Leissa, 1995; Tauchert, 1987, 1991). The set of membrane stresses is regarded as the initial stresses and is introduced into the variational equations (Bolotin, 1964). The incremental stresses associated with the small incremental displacements perturbed from the state of neutral equilibrium will be considered.

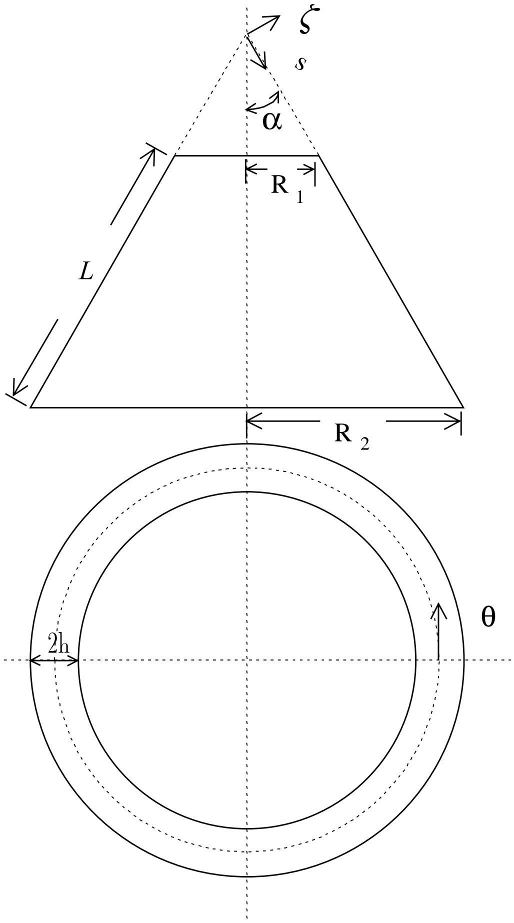


Fig. 1. The geometry and coordinate system for a laminated conical shell.

Referring to the configuration of the initial membrane state of stress, the motion equations are given by (Saada, 1974; Soedel, 1993)

$$h_\theta\sigma_{s,s} + s_\alpha\sigma_s - s_\alpha\sigma_\theta + \tau_{s\theta,\theta} + c_\alpha\tau_{s\zeta} + h_\theta\tau_{s\zeta,\zeta} - \sigma_s^t(h_\theta u_{s,s})_{,s} - (\sigma_\theta^t/h_\theta)\left[u_{s,\theta\theta} - s_\alpha^2 u_s - 2s_\alpha u_{\theta,\theta} - s_\alpha c_\alpha u_\zeta\right] - \tau_{s\theta}^t[2u_{s,s\theta} - 2s_\alpha u_{\theta,s}] = \rho h_\theta u_{s,tt}, \quad (1)$$

$$2s_\alpha\tau_{s\theta} + h_\theta\tau_{s\theta,s} + \sigma_{\theta,\theta} + 2c_\alpha\tau_{\theta\zeta} + h_\theta\tau_{\theta\zeta,\zeta} - \sigma_s^t(h_\theta u_{\theta,s})_{,s} - (\sigma_\theta^t/h_\theta)[2s_\alpha u_{s,\theta} + u_{\theta,\theta\theta} - u_\theta + 2c_\alpha u_{\zeta,\theta}] - \tau_{s\theta}^t[2s_\alpha u_{s,s} + 2u_{\theta,s\theta} + 2c_\alpha u_{\zeta,s}] = \rho h_\theta u_{\theta,tt}, \quad (2)$$

$$-c_\alpha\sigma_\theta + s_\alpha\tau_{s\zeta} + h_\theta\tau_{s\zeta,s} + \tau_{\theta\zeta,\theta} + c_\alpha\sigma_\zeta + h_\theta\sigma_{\zeta,\zeta} - \sigma_s^t(h_\theta u_{\zeta,s})_{,s} - (\sigma_\theta^t/h_\theta)\left[-s_\alpha c_\alpha u_s - 2c_\alpha u_{\theta,\theta} + u_{\zeta,\theta\theta} - c_\alpha^2 u_\zeta\right] - \tau_{s\theta}^t[-2c_\alpha u_{\theta,s} + 2u_{\zeta,s\theta}] = \rho h_\theta u_{\zeta,tt}, \quad (3)$$

where $\rho$ is the mass density; $h_\theta = ss_\alpha + \zeta c_\alpha$, $s_\alpha = \sin\alpha$ and $c_\alpha = \cos\alpha$; $\sigma_s$, $\sigma_\theta$, $\sigma_\zeta$, $\tau_{s\zeta}$, $\tau_{\theta\zeta}$ and $\tau_{s\theta}$ are the incremental stresses; $u_s$, $u_\theta$ and $u_\zeta$ are the incremental displacements; the commas stand for differentiation with respect to the suffix variables; $\sigma_s^t$, $\sigma_\theta^t$ and $\tau_{s\theta}^t$ denote the initial membrane stresses due to a temperature change $\Delta T$.

In the present study, the temperature change $\Delta T$ is considered a periodic function in time and a certain distributed function in the thickness coordinate

$$\Delta T = \alpha_s \Delta T_{cr} \phi(\zeta) + \alpha_d \Delta T_{cr} \phi(\zeta) \cos \Omega t, \tag{4}$$

where $\alpha_s$ and $\alpha_d$ are the static and dynamic parameters related to the critical value of the temperature field in the static thermoelastic buckling problem (Wu and Chiu, 2001); $\Omega$ is the angular frequency of the temperature field. The time variable is represented as $t$, and the distributed temperature function $\phi(\zeta)$ is normalized as $\int_{-h}^{h} \phi^2(\zeta)\,d\zeta = 1$.

The corresponding relations between $(\sigma_s^t, \sigma_\theta^t, \tau_{s\theta}^t)$ and $\Delta T$ for a monoclinic material are $\sigma_s^t = \eta_s \Delta T$, $\sigma_\theta^t = \eta_\theta \Delta T$, $\tau_{s\theta}^t = \eta_{s\theta} \Delta T$ and

$$\begin{Bmatrix} \eta_s \\ \eta_\theta \\ \eta_{s\theta} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{26} \\ c_{16} & c_{26} & c_{36} & c_{66} \end{bmatrix} \begin{Bmatrix} c_\theta^2 \alpha_1 + s_\theta^2 \alpha_2 \\ s_\theta^2 \alpha_1 + c_\theta^2 \alpha_2 \\ \alpha_3 \\ 2 c_\theta s_\theta (\alpha_1 - \alpha_2) \end{Bmatrix}, \tag{5}$$

where $c_{ij}$ denote the stiffness coefficients; $\alpha_1$, $\alpha_2$ and $\alpha_3$ are the coefficients of thermal expansion along principal material axes; $c_\theta, s_\theta = (\cos, \sin)\theta$.

The incremental stress–strain relations for a monoclinic material are given by (Saada, 1974)

$$\begin{Bmatrix} \sigma_s \\ \sigma_\theta \\ \sigma_\zeta \\ \tau_{\theta\zeta} \\ \tau_{s\zeta} \\ \tau_{s\theta} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \varepsilon_\zeta \\ \gamma_{\theta\zeta} \\ \gamma_{s\zeta} \\ \gamma_{s\theta} \end{Bmatrix}, \tag{6}$$

where $\varepsilon_s$, $\varepsilon_\theta$, $\varepsilon_\zeta$, $\gamma_{s\zeta}$, $\gamma_{\theta\zeta}$, $\gamma_{s\theta}$ are the incremental strain components.

The kinematics relations between the incremental strains and incremental displacements are written as (Saada, 1974)

$$\begin{Bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \varepsilon_\zeta \\ \gamma_{s\zeta} \\ \gamma_{\theta\zeta} \\ \gamma_{s\theta} \end{Bmatrix} = \begin{bmatrix} \partial_s & 0 & 0 \\ s_\alpha/h_\theta & \partial_\theta/h_\theta & c_\alpha/h_\theta \\ 0 & 0 & \partial_\zeta \\ \partial_\zeta & 0 & \partial_s \\ 0 & \partial_\zeta - (c_\alpha/h_\theta) & \partial_\theta/h_\theta \\ \partial_\theta/h_\theta & \partial_s - (s_\alpha/h_\theta) & 0 \end{bmatrix} \begin{Bmatrix} u_s \\ u_\theta \\ u_\zeta \end{Bmatrix}, \tag{7}$$

in which $\partial_s = \partial/\partial s$, $\partial_\theta = \partial/\partial\theta$, $\partial_\zeta = \partial/\partial\zeta$.

## 3. Nondimensionalization and asymptotic expansion

A set of dimensionless field variables is used in the present formulation and defined as follows:

$$x_1 = s/R\varepsilon, \quad x_2 = \theta, \quad x_3 = \zeta/h, \tag{8a–c}$$

$$u_1 = u_s/R\varepsilon, \quad u_2 = u_\theta/R\varepsilon, \quad u_3 = u_\zeta/R, \tag{8d–f}$$

$$\sigma_1 = \sigma_s/Q, \quad \sigma_2 = \sigma_\theta/Q, \quad \tau_{12} = \tau_{s\theta}/Q, \tag{8g–i}$$

$$\tau_{13} = \tau_{s\zeta}/Q\varepsilon, \quad \tau_{23} = \tau_{\theta\zeta}/Q\varepsilon, \tag{8j, k}$$

$$\sigma_3 = \sigma_\zeta / Q\varepsilon^2, \tag{8l}$$

$$\Delta\widetilde{T} = \eta\Delta T / Q\varepsilon^2, \tag{8m}$$

$$\widetilde{\eta}_1 = \eta_s/\eta, \quad \widetilde{\eta}_2 = \eta_\theta/\eta, \quad \widetilde{\eta}_6 = \eta_{s\theta}/\eta, \tag{8n–p}$$

where $\varepsilon^2 = h/R$ is a small parameter, usually much less than 1; $R$ denotes a characteristic length of the shell; $\eta$ and $Q$ are reference thermoelastic moduli; $\Delta\widetilde{T} = \widetilde{T}\widetilde{\phi}(x_3)$ in which $\widetilde{T} = (\alpha_s\Delta T_{cr} + \alpha_d\Delta T_{cr}\cos\Omega t)\eta/Q\varepsilon^2$ and $\widetilde{\phi}(x_3) = \phi(\zeta)$.

The dimensionless multiple time scales are introduced in the formulation and defined in the following form

$$\tau_k = \varepsilon^{2k}\sqrt{Q/\rho_0}\,t/R \quad (k = 0, 1, 2, \ldots), \tag{9}$$

where $\rho_0$ is the reference mass density.

The increments of the displacements $(u_s, u_\theta, u_\zeta)$ and transverse stresses $(\tau_{s\zeta}, \tau_{\theta\zeta}, \sigma_\zeta)$ are regarded as the primary field variables. After eliminating the in-surface stresses $\sigma_s$, $\sigma_\theta$ and $\tau_{s\theta}$ from (1)–(7) we reformulate the 3D equations of motion in the dimensionless form as

$$u_{3,3} = -\varepsilon^2\mathbf{L}_1\mathbf{u} - \varepsilon^2\tilde{l}_{33}u_3 + \varepsilon^4(Q/c_{33})\sigma_3, \tag{10}$$

$$\mathbf{u}_{,3} = -\mathbf{D}u_3 + \varepsilon^2\mathbf{L}_2\mathbf{u} + \varepsilon^2\mathbf{S}\boldsymbol{\sigma}_s + \varepsilon^4\mathbf{L}_3\boldsymbol{\sigma}_s, \tag{11}$$

$$\begin{aligned}\boldsymbol{\sigma}_{s,3} = {} & -\mathbf{L}_4\mathbf{u} - \mathbf{L}_5u_3 - \varepsilon^2\mathbf{L}_6\boldsymbol{\sigma}_s - \varepsilon^2\mathbf{L}_7\sigma_3 + \varepsilon^2(\mathbf{T}_1\mathbf{u} + \mathbf{T}_2u_3)\Delta\widetilde{T} \\ & + \rho_1\left[\frac{\partial^2}{\partial\tau_0^2} + 2\varepsilon^2\frac{\partial^2}{\partial\tau_0\,\partial\tau_1} + \varepsilon^4\left(2\frac{\partial^2}{\partial\tau_0\,\partial\tau_2} + \frac{\partial^2}{\partial\tau_1^2}\right) + \cdots\right]\mathbf{u},\end{aligned} \tag{12}$$

$$\begin{aligned}\sigma_{3,3} = {} & \mathbf{L}_8\mathbf{u} + \tilde{l}_{63}u_3 - \mathbf{D}^{\mathrm{T}}\boldsymbol{\sigma}_s - \tau_{13}s_\alpha/r - \varepsilon^2\tilde{l}_{64}\tau_{13} - \varepsilon^2\tilde{l}_{65}\sigma_3 + \left[\tilde{t}_{34}u_3 + \varepsilon^2\left(\mathbf{T}_3\mathbf{u} + \tilde{t}_{33}u_3\right)\right]\Delta\widetilde{T} \\ & + \rho_2\left[\frac{\partial^2}{\partial\tau_0^2} + 2\varepsilon^2\frac{\partial^2}{\partial\tau_0\,\partial\tau_1} + \varepsilon^4\left(2\frac{\partial^2}{\partial\tau_0\,\partial\tau_2} + \frac{\partial^2}{\partial\tau_1^2}\right) + \cdots\right]u_3,\end{aligned} \tag{13}$$

$$\boldsymbol{\sigma}_m = \mathbf{L}_9\mathbf{u} + \mathbf{L}_{10}u_3 + \varepsilon^2\mathbf{L}_{11}\sigma_3, \tag{14}$$

where

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \boldsymbol{\sigma}_s = \begin{Bmatrix} \tau_{13} \\ \tau_{23} \end{Bmatrix}, \quad \boldsymbol{\sigma}_m = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \tilde{l}_{13} \\ \tilde{l}_{23} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \tilde{l}_{14} & \tilde{l}_{15} \\ \tilde{l}_{15} & \tilde{l}_{25} \end{bmatrix}, \quad \mathbf{L}_1 = \begin{bmatrix} \tilde{l}_{31} & \tilde{l}_{32} \end{bmatrix},$$

$$\mathbf{L}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{l}_{22} \end{bmatrix}, \quad \mathbf{L}_3 = \begin{bmatrix} 0 & 0 \\ \tilde{l}_{26} & \tilde{l}_{27} \end{bmatrix}, \quad \mathbf{L}_4 = \begin{bmatrix} \tilde{l}_{41} & \tilde{l}_{42} \\ \tilde{l}_{51} & \tilde{l}_{52} \end{bmatrix}, \quad \mathbf{L}_5 = \begin{bmatrix} \tilde{l}_{43} \\ \tilde{l}_{53} \end{bmatrix}, \quad \mathbf{L}_6 = \begin{bmatrix} \tilde{l}_{44} & 0 \\ 0 & \tilde{l}_{55} \end{bmatrix},$$

$$\mathbf{L}_7 = \begin{bmatrix} \tilde{l}_{46} \\ \tilde{l}_{56} \end{bmatrix}, \quad \mathbf{L}_8 = \begin{bmatrix} \tilde{l}_{61} & \tilde{l}_{62} \end{bmatrix}, \quad \mathbf{L}_9 = \begin{bmatrix} \tilde{l}_{71} & \tilde{l}_{72} \\ \tilde{l}_{81} & \tilde{l}_{82} \\ \tilde{l}_{91} & \tilde{l}_{92} \end{bmatrix}, \quad \mathbf{L}_{10} = \begin{bmatrix} \tilde{l}_{73} \\ \tilde{l}_{83} \\ \tilde{l}_{93} \end{bmatrix}, \quad \mathbf{L}_{11} = \begin{bmatrix} \tilde{c}_{13} \\ \tilde{c}_{23} \\ \tilde{c}_{36} \end{bmatrix},$$

$$\mathbf{T}_1 = \begin{bmatrix} \tilde{t}_{11} & \tilde{t}_{12} \\ \tilde{t}_{21} & \tilde{t}_{22} \end{bmatrix}, \quad \mathbf{T}_2 = \begin{bmatrix} \tilde{t}_{13} \\ \tilde{t}_{23} \end{bmatrix}, \quad \mathbf{T}_3 = \begin{bmatrix} \tilde{t}_{31} & \tilde{t}_{32} \end{bmatrix}, \quad r = x_1s_\alpha,$$

$\rho_1 = (\rho/\rho_0)(h/R)\gamma_\theta, \quad \rho_2 = (\rho/\rho_0)\gamma_\theta.$

The detailed expressions of $\tilde{l}_{ij}$ in the matrices $\mathbf{L}_i$ ($i = 1$–11) and $\tilde{t}_{ij}$ in $\mathbf{T}_i$ ($i = 1$–3) can be found in an early paper (Wu and Chiu, 2001).

Noting that (10)–(14) contain only even power terms of $\varepsilon$, we expand the displacements and stresses as

$$f(x_1,x_2,x_3,\tau_0,\tau_1,\ldots,\varepsilon) = f^{(0)}(x_1,x_2,x_3,\tau_0,\tau_1,\ldots) + \varepsilon^2 f^{(1)}(x_1,x_2,x_3,\tau_0,\tau_1,\ldots) + \varepsilon^4 f^{(2)}(x_1,x_2,x_3,\tau_0,\tau_1,\ldots) + \cdots, \tag{15}$$

where $f = \sigma_{ij},\ u_i$.

After substituting (15) into (10)–(14) and collecting coefficients of equal powers of $\varepsilon$, we obtain the following sets of equations for various orders.

Order $\varepsilon^0$:

$$u_{3,3}^{(0)} = 0, \tag{16}$$

$$\mathbf{u}_{,3}^{(0)} = -\mathbf{D}u_3^{(0)}, \tag{17}$$

$$\boldsymbol{\sigma}_{s,3}^{(0)} = -\mathbf{L}_4\mathbf{u}^{(0)} - \mathbf{L}_5 u_3^{(0)} + \rho_1 \frac{\partial^2 \mathbf{u}^{(0)}}{\partial \tau_0^2}, \tag{18}$$

$$\boldsymbol{\sigma}_{3,3}^{(0)} = \mathbf{L}_8\mathbf{u}^{(0)} + \tilde{l}_{63}u_3^{(0)} - \mathbf{D}^{\mathrm{T}}\boldsymbol{\sigma}_s^{(0)} - \tau_{13}^{(0)}s_\alpha/r + \tilde{t}_{34}u_3^{(0)}\Delta\widetilde{T} + \rho_2 \frac{\partial^2 u_3^{(0)}}{\partial \tau_0^2}, \tag{19}$$

$$\boldsymbol{\sigma}_m^{(0)} = \mathbf{L}_9\mathbf{u}^{(0)} + \mathbf{L}_{10}u_3^{(0)}. \tag{20}$$

Order $\varepsilon^{2k}$ ($k = 1,2,3,\ldots$):

$$u_{3,3}^{(k)} = -\mathbf{L}_1\mathbf{u}^{(k-1)} - \tilde{l}_{33}u_3^{(k-1)} + (Q/c_{33})\sigma_3^{(k-2)}, \tag{21}$$

$$\mathbf{u}_{,3}^{(k)} = -\mathbf{D}u_3^{(k)} + \mathbf{L}_2\mathbf{u}^{(k-1)} + \mathbf{S}\boldsymbol{\sigma}_s^{(k-1)} + \mathbf{L}_3\boldsymbol{\sigma}_s^{(k-2)}, \tag{22}$$

$$\boldsymbol{\sigma}_{s,3}^{(k)} = -\mathbf{L}_4\mathbf{u}^{(k)} - \mathbf{L}_5u_3^{(k)} - \mathbf{L}_6\boldsymbol{\sigma}_s^{(k-1)} - \mathbf{L}_7\sigma_3^{(k-1)} + \left[\mathbf{T}_1\mathbf{u}^{(k-1)} + \mathbf{T}_2u_3^{(k-1)}\right]\Delta\widetilde{T} + \rho_1\left[\frac{\partial^2\mathbf{u}^{(k)}}{\partial\tau_0^2} + 2\frac{\partial^2\mathbf{u}^{(k-1)}}{\partial\tau_0\,\partial\tau_1} + \left(2\frac{\partial^2\mathbf{u}^{(k-2)}}{\partial\tau_0\,\partial\tau_1} + \frac{\partial^2\mathbf{u}^{(k-2)}}{\partial\tau_1^2}\right) + \cdots\right], \tag{23}$$

$$\sigma_{3,3}^{(k)} = \mathbf{L}_8\mathbf{u}^{(k)} + \tilde{l}_{63}u_3^{(k)} - \mathbf{D}^{\mathrm{T}}\boldsymbol{\sigma}_s^{(k)} - \tau_{13}^{(k)}s_\alpha/r - \tilde{l}_{64}\tau_{13}^{(k-1)} - \tilde{l}_{65}\sigma_3^{(k-1)} + \tilde{t}_{34}u_3^{(k)}\Delta\widetilde{T} + \left[\mathbf{T}_3\mathbf{u}^{(k-1)} + \tilde{t}_{33}u_3^{(k-1)}\right]\Delta\widetilde{T} + \rho_2\left[\frac{\partial^2u_3^{(k)}}{\partial\tau_0^2} + 2\frac{\partial^2u_3^{(k-1)}}{\partial\tau_0\,\partial\tau_1} + \left(2\frac{\partial^2u_3^{(k-2)}}{\partial\tau_0\,\partial\tau_1} + \frac{\partial^2u_3^{(k-2)}}{\partial\tau_1^2}\right) + \cdots\right], \tag{24}$$

$$\boldsymbol{\sigma}_m^{(k)} = \mathbf{L}_9\mathbf{u}^{(k)} + \mathbf{L}_{10}u_3^{(k)} + \mathbf{L}_{11}\sigma_3^{(k-1)}, \tag{25}$$

where displacement and stress components $f^{(j)} = 0$ for $j < 0$.

The associated dimensionless boundary conditions for various orders are described as follows: on the inner and outer surfaces the following traction conditions must be satisfied:

$$\left[\tau_{13}^{(k)}, \tau_{23}^{(k)}, \sigma_3^{(k)}\right] = [0,0,0] \quad (k = 0,1,2,\ldots) \text{ on } x_3 = \pm 1. \tag{26}$$

Along the edges at $x_1 = 0$ and $L/\sqrt{Rh}$, one member of each pair of the following quantities must be satisfied:

$$n_1\sigma_1^{(0)} + n_2\tau_{12}^{(0)} = \bar{p}_1, \quad \text{or} \quad u_1^{(0)} = \bar{u}_1, \tag{27a}$$

$$n_1\tau_{12}^{(0)} + n_2\sigma_2^{(0)} = \bar{p}_2, \quad \text{or} \quad u_2^{(0)} = \bar{u}_2, \tag{27b}$$

$$n_1\sigma_{13}^{(0)} + n_2\tau_{23}^{(0)} = \bar{p}_3, \quad \text{or} \quad u_3^{(0)} = \bar{u}_3 \tag{27c}$$

and

$$n_1\sigma_1^{(k)} + n_2\tau_{12}^{(k)} = 0, \quad \text{or} \quad u_1^{(k)} = 0 \ \ (k = 1, 2, 3, \ldots), \tag{28a}$$

$$n_1\tau_{12}^{(k)} + n_2\sigma_2^{(k)} = 0, \quad \text{or} \quad u_2^{(k)} = 0 \ \ (k = 1, 2, 3, \ldots), \tag{28b}$$

$$n_1\tau_{13}^{(k)} + n_2\tau_{23}^{(k)} = 0, \quad \text{or} \quad u_3^{(k)} = 0 \ \ (k = 1, 2, 3, \ldots), \tag{28c}$$

where $\bar{p}_1 = \bar{p}_s/Q$, $\bar{p}_2 = \bar{p}_\theta/Q$, $\bar{p}_3 = \bar{p}_\zeta/Q\varepsilon$; $\bar{u}_1 = \bar{u}_s/\sqrt{Rh}$, $\bar{u}_2 = \bar{u}_\theta$ and $\bar{u}_3 = \bar{u}_\zeta/R$. ($\bar{p}_s$, $\bar{p}_\theta$, $\bar{p}_\zeta$) and ($\bar{u}_s$, $\bar{u}_\theta$, $\bar{u}_\zeta$) are prescribed traction and displacement components, respectively.

## 4. Asymptotic formulation

### *4.1. The leading-order level*

Integrating the asymptotic equations (16)–(19) and applying the lateral boundary conditions (26) at the inner surfaces ($x_3 = -1$), we obtain

$$u_3^{(0)} = u_3^0(x_1, x_2), \tag{29}$$

$$\mathbf{u}^{(0)} = \mathbf{u}^0(x_1, x_2) - x_3\mathbf{D}u_3^0, \tag{30}$$

$$\boldsymbol{\sigma}_s^{(0)} = -\int_{-1}^{x_3} \left[\mathbf{L}_4\left(\mathbf{u}^0 - \eta\mathbf{D}u_3^0\right) + \mathbf{L}_5 u_3^0\right] \mathrm{d}\eta + \frac{\partial^2}{\partial\tau_0^2}\left[\int_{-1}^{x_3} \rho_1(\mathbf{u}^0 - \eta\mathbf{D}u_3^0)\, \mathrm{d}\eta\right], \tag{31}$$

$$\begin{aligned}\sigma_3^{(0)} = {} & \int_{-1}^{x_3} [\mathbf{L}_8\left(\mathbf{u}^0 - \eta\mathbf{D}u_3^0\right) + \tilde{l}_{63}u_3^0]\, \mathrm{d}\eta + \int_{-1}^{x_3} \left\{(x_3 - \eta)\mathbf{D}^{\mathrm{T}}\left[\mathbf{L}_4\left(\mathbf{u}^0 - \eta\mathbf{D}u_3^0\right) + \mathbf{L}_5 u_3^0\right]\right\} \mathrm{d}\eta + (s_\alpha/r) \\ & \times \int_{-1}^{x_3} \left\{(x_3 - \eta)\left[\mathbf{L}_{12}\left(\mathbf{u}^0 - \eta\mathbf{D}u_3^0\right) + \tilde{l}_{43}u_3^0\right]\right\} \mathrm{d}\eta - \int_{-1}^{x_3} \left(\tilde{t}_{34}u_3^0\Delta\widetilde{T}\right) \mathrm{d}\eta + \left(\int_{-1}^{x_3} \rho_2\, \mathrm{d}\eta\right) \frac{\partial^2 u_3^0}{\partial\tau_0^2} - \frac{\partial^2}{\partial\tau_0^2} \\ & \times \left[\int_{-1}^{x_3} (x_3 - \eta)\rho_1\mathbf{D}^{\mathrm{T}}\left(\mathbf{u}^0 - \eta\mathbf{D}u_3^0\right) \mathrm{d}\eta\right] - \frac{\partial^2}{\partial\tau_0^2}\left[\int_{-1}^{x_3} (x_3 - \eta)\rho_1(s_\alpha/r)\left(u_1^0 - \eta u_{3,1}^0\right) \mathrm{d}\eta\right],\end{aligned} \tag{32}$$

where $u_3^0(x_1, x_2)$, $\mathbf{u}^0 = \left\{ u_1^0(x_1, x_2) \quad u_2^0(x_1, x_2) \right\}^{\mathrm{T}}$ represent the middle surface displacements; $\mathbf{L}_{12} = \left[\, \tilde{l}_{41} \quad \tilde{l}_{42} \,\right]$ in which the detailed expressions of $\tilde{l}_{41}$ and $\tilde{l}_{42}$ can also be found in an early paper (Wu and Chiu, 2001).

Imposing the lateral boundary conditions (26) at the outer surface ($x_3 = 1$) on (31) and (32) and simplifying the resulting equations, we can rewrite (31) and (32) as

$$K_{11}u_1^0 + K_{12}u_2^0 + K_{13}u_3^0 = I_{11}\frac{\partial^2}{\partial\tau_0^2}(u_{3,1}^0) - I_{10}\frac{\partial^2 u_1^0}{\partial\tau_0^2}, \tag{33}$$

$$K_{21}u_1^0 + K_{22}u_2^0 + K_{23}u_3^0 = (I_{11}/r)\frac{\partial^2}{\partial\tau_0^2}(u_{3,2}^0) - I_{10}\frac{\partial^2 u_2^0}{\partial\tau_0^2}, \tag{34}$$

$$K_{31}u_1^0 + K_{32}u_2^0 + (K_{33} + K_N\widetilde{T})u_3^0 = I_{12}\frac{\partial^2}{\partial\tau_0^2}(\mathbf{D}^{\mathrm{T}}\mathbf{D}u_3^0 + s_\alpha u_{3,1}^0/r) - I_{20}\frac{\partial^2 u_3^0}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}(\mathbf{D}^{\mathrm{T}}\mathbf{u}^0 + s_\alpha u_1^0/r), \tag{35}$$

where $K_{ij}$ are the differential operators. For brevity, the operators for cross-ply laminated shells are given by

$$K_{11} = -(\widehat{A}_{11}\partial_{11} + \bar{A}_{66}\partial_{22}/r^2 + \widetilde{A}_{11}s_\alpha\partial_1/r - \bar{A}_{22}s_\alpha^2/r^2),$$

$$K_{12} = -\left[(\widetilde{A}_{12} + \widetilde{A}_{66})\partial_{12}/r - (\bar{A}_{22} + \bar{A}_{66})s_\alpha\partial_2/r^2\right],$$

$$\begin{aligned}K_{13} = {} & \widehat{B}_{11}\partial_{111} + (\widetilde{B}_{12} + \overline{B}_{66} + \widetilde{B}_{66})\partial_{122}/r^2 + \widetilde{B}_{11}s_\alpha\partial_{11}/r - (\widetilde{B}_{12} + \overline{B}_{22} + \widetilde{B}_{66} + \overline{B}_{66})s_\alpha\partial_{22}/r^3 \\ & - (\widetilde{A}_{12}c_\alpha\sqrt{R/h}/r + \overline{B}_{22}s_\alpha^2/r^2)\partial_1 + \bar{A}_{22}s_\alpha c_\alpha\sqrt{R/h}/r^2,\end{aligned}$$

$$K_{21} = -\left[(\widetilde{A}_{12} + \widetilde{A}_{66})\partial_{12}/r + (\bar{A}_{22} + \bar{A}_{66})s_\alpha\partial_2/r^2\right],$$

$$K_{22} = -\left(\widehat{A}_{66}\partial_{11} + \bar{A}_{22}\partial_{22}/r^2 + \widetilde{A}_{66}s_\alpha\partial_1/r - \bar{A}_{66}s_\alpha^2/r^2\right),$$

$$\begin{aligned}K_{23} = {} & (\widetilde{B}_{12} + \widehat{B}_{66} + \widetilde{B}_{66})\partial_{112}/r + \overline{B}_{22}\partial_{222}/r^3 + (\overline{B}_{22} + \overline{B}_{66} + \widetilde{B}_{66} - 2\widehat{B}_{66})s_\alpha\partial_{12}/r^2 \\ & - \left[\bar{A}_{22}c_\alpha\sqrt{R/h}/r^2 + (\widetilde{B}_{66} + \overline{B}_{66} - 2\widehat{B}_{66})s_\alpha^2/r^3\right]\partial_2,\end{aligned}$$

$$\begin{aligned}K_{31} = {} & -\widehat{B}_{11}\partial_{111} - (\widetilde{B}_{12} + \overline{B}_{66} + \widetilde{B}_{66})\partial_{122}/r^2 - (\widetilde{B}_{11} + \widehat{B}_{11})s_\alpha\partial_{11}/r - \overline{B}_{22}s_\alpha\partial_{22}/r^3 \\ & + (\widetilde{A}_{12}c_\alpha\sqrt{R/h}/r + \overline{B}_{22}s_\alpha^2/r^2)\partial_1 + \bar{A}_{22}s_\alpha c_\alpha\sqrt{R/h}/r^2 - \overline{B}_{22}s_\alpha^3/r^3 - \widehat{B}_{11,1}\partial_{11} + \overline{B}_{22,1}s_\alpha^2/r^2 - \overline{B}_{66,1}\partial_{22}/r^2,\end{aligned}$$

$$\begin{aligned}K_{32} = {} & -(\widetilde{B}_{12} + \widehat{B}_{66} + \widetilde{B}_{66})\partial_{112}/r - \overline{B}_{22}\partial_{222}/r^3 + (\overline{B}_{22} + \overline{B}_{66} - \widetilde{B}_{66})s_\alpha\partial_{12}/r^2 \\ & + (\bar{A}_{22}c_\alpha\sqrt{R/h}/r^2 - \overline{B}_{22}s_\alpha^2/r^3)\partial_2 + (\overline{B}_{22} + \overline{B}_{66})_{,1}\, s_\alpha\partial_2/r^2,\end{aligned}$$

$$\begin{aligned}K_{33} = {} & \widehat{D}_{11}\partial_{1111} + (2\widetilde{D}_{12} + \overline{D}_{66} + 2\widetilde{D}_{66} + \widehat{D}_{66})\partial_{1122}/r^2 + \overline{D}_{22}\partial_{2222}/r + (\widetilde{D}_{11} + \widehat{D}_{11})s_\alpha\partial_{1111}/r \\ & - (2\widehat{D}_{66} + \widetilde{D}_{66} + \overline{D}_{66} + 2\widetilde{D}_{12})s_\alpha\partial_{122}/r^3 - (2\widetilde{B}_{12}c_\alpha\sqrt{R/h}/r + \overline{D}_{22}s_\alpha^2/r^2)\partial_{11} - \Big[2\overline{B}_{22}c_\alpha\sqrt{R/h}/r^3 \\ & \quad - (2\widetilde{D}_{12} + 2\overline{D}_{22} + \widetilde{D}_{66} + \overline{D}_{66} + 2\widehat{D}_{66})s_\alpha^2/r^4\Big]\partial_{22} + \overline{D}_{22}s_\alpha^3\partial_1/r^3 + \bar{A}_{22}(c_\alpha\sqrt{R/h}/r)^2 - \overline{B}_{22}s_\alpha^2c_\alpha\sqrt{R/h}/r^3 \\ & + \widehat{D}_{11,1}\partial_{1111} + \overline{D}_{66,1}\partial_{122}/r^2 - s_\alpha(\overline{D}_{22} + \overline{D}_{66})_{,1}\,\partial_{22}/r^3 - s_\alpha^2\overline{D}_{22,1}\partial_1/r^2 + s_\alpha c_\alpha\sqrt{R/h}\overline{B}_{22,1}/r^2,\end{aligned}$$

$$K_N = \hat{\gamma}_1\partial_{11} + (\bar{\gamma}_2/r^2)\partial_{22} + (\tilde{\gamma}_1 s_\alpha/r)\partial_1,$$

$$I_{10} = \int_{-1}^{1}\rho_1\,\mathrm{d}x_3, \quad I_{11} = \int_{-1}^{1}\rho_1 x_3\,\mathrm{d}x_3, \quad I_{12} = \int_{-1}^{1}\rho_1 x_3^2\,\mathrm{d}x_3, \quad I_{20} = \int_{-1}^{1}\rho_2\,\mathrm{d}x_3,$$

$$\left[\widehat{A}_{ij} \quad \widetilde{A}_{ij} \quad \bar{A}_{ij}\right] = \int_{-1}^{1}\widetilde{Q}_{ij}\left[\gamma_\theta \quad 1 \quad (1/\gamma_\theta)\right]\mathrm{d}x_3,$$

$$[\widehat{B}_{ij} \quad \widetilde{B}_{ij} \quad \overline{B}_{ij}] = \int_{-1}^{1} \widetilde{Q}_{ij} x_3 [\gamma_\theta \quad 1 \quad (1/\gamma_\theta)]\, \mathrm{d}x_3,$$

$$[\widehat{D}_{ij} \quad \widetilde{D}_{ij} \quad \overline{D}_{ij}] = \int_{-1}^{1} \widetilde{Q}_{ij} x_3^2 [\gamma_\theta \quad 1 \quad (1/\gamma_\theta)]\, \mathrm{d}x_3,$$

$$\left[\hat{\gamma}_{ij} \quad \widetilde{\gamma}_{ij} \quad \overline{\gamma}_{ij}\right] = \int_{-1}^{1} \widetilde{\eta}_{ij} \widetilde{\phi} [\gamma_\theta \quad 1 \quad (1/\gamma_\theta)]\, \mathrm{d}x_3.$$

### 4.2. The higher-order levels

Integrating (21)–(24), we obtain

$$u_3^{(k)} = u_3^k(x_1, x_2) + \phi_{3k}(x_1, x_2, x_3), \tag{36}$$

$$\mathbf{u}^{(k)} = \mathbf{u}^k(x_1, x_2) - x_3 \mathbf{D} u_3^k + \boldsymbol{\phi}_k(x_1, x_2, x_3), \tag{37}$$

$$\boldsymbol{\sigma}_s^{(k)} = -\int_{-1}^{x_3} \left[\mathbf{L}_4(\mathbf{u}^k - \eta \mathbf{D} u_3^k) + \mathbf{L}_5 u_3^k\right] \mathrm{d}\eta - \mathbf{f}_k(x_1, x_2, x_3), \tag{38}$$

$$\begin{aligned}\sigma_3^{(k)} = {} & \int_{-1}^{x_3} \left[\mathbf{L}_8\left(\mathbf{u}^k - \eta \mathbf{D} u_3^k\right) + \widetilde{l}_{63} u_3^k\right] \mathrm{d}\eta + \int_{-1}^{x_3} \left\{(x_3 - \eta) \mathbf{D}^{\mathrm{T}} \left[\mathbf{L}_4\left(\mathbf{u}^k - \eta \mathbf{D} u_3^k\right) + \mathbf{L}_5 u_3^k\right]\right\} \mathrm{d}\eta + (s_\alpha/r) \\ & \times \int_{-1}^{x_3} \left\{(x_3 - \eta)\left[\mathbf{L}_{12}\left(\mathbf{u}^k - \eta \mathbf{D} u_3^k\right) + \widetilde{l}_{43} u_3^k\right]\right\} \mathrm{d}\eta - f_{3k}(x_1, x_2, x_3),\end{aligned} \tag{39}$$

where $u_3^k$ and $\mathbf{u}^k$ are the higher-order modifications to middle surface displacements. The relevant functions are given by

$$\mathbf{u}^k = \left\{ u_1^k(x_1, x_2) \quad u_2^k(x_1, x_2) \right\}^{\mathrm{T}},$$

$$\begin{aligned} f_{3k}(x_1, x_2, x_3) = {} & -\int_{-1}^{x_3} \left[\mathbf{D}^{\mathrm{T}} \mathbf{f}_k + s_\alpha f_{1k}/r + \mathbf{L}_8 \boldsymbol{\phi}_k + \tilde{l}_{63} \phi_{3k} - \tilde{l}_{64} \tau_{13}^{(k-1)} - \tilde{l}_{65} \sigma_3^{(k-1)}\right] \mathrm{d}\eta \\ & + \int_{-1}^{x_3} \left[\tilde{t}_{34} \phi_{3k} + \mathbf{T}_3 \mathbf{u}^{(k-1)} + \tilde{t}_{33} u_3^{(k-1)}\right] \Delta \widetilde{T}\, \mathrm{d}\eta - \left[\frac{\partial^2}{\partial \tau_0^2}\left(\int_{-1}^{x_3} \rho_2 \phi_{3k}\, \mathrm{d}\eta\right)\right. \\ & \left. + \frac{\partial^2}{\partial \tau_0 \partial \tau_1}\left(2\int_{-1}^{x_3} \rho_2 u_3^{(k-1)}\, \mathrm{d}\eta\right) + \cdots \right], \end{aligned}$$

$$\begin{aligned} \mathbf{f}_k = \begin{Bmatrix} f_{1k}(x_1, x_2, x_3) \\ f_{2k}(x_1, x_2, x_3) \end{Bmatrix} = {} & \int_{-1}^{x_3} \left[\mathbf{L}_4 \boldsymbol{\phi}_k + \mathbf{L}_5 \phi_{3k} + \mathbf{L}_6 \boldsymbol{\sigma}_s^{(k-1)} + \mathbf{L}_7 \sigma_3^{(k-1)}\right] \mathrm{d}\eta - \int_{-1}^{x_3} \left[\mathbf{T}_1 \mathbf{u}^{(k-1)} + \mathbf{T}_2 u_3^{(k-1)}\right] \Delta \widetilde{T}\, \mathrm{d}\eta \\ & - \left[\frac{\partial^2}{\partial \tau_0^2}\left(\int_{-1}^{x_3} \rho_1 \boldsymbol{\phi}_k\, \mathrm{d}\eta\right) + \frac{\partial^2}{\partial \tau_0\, \partial \tau_1}\left(2\int_{-1}^{x_3} \rho_1 \mathbf{u}^{(k-1)}\, \mathrm{d}\eta\right) + \cdots \right], \end{aligned}$$

$$\phi_{3k}(x_1, x_2, x_3) = -\int_{0}^{x_3} \left[\mathbf{L}_1 \mathbf{u}^{(k-1)} + \tilde{l}_{33} u_3^{(k-1)} - (Q/c_{33}) \sigma_3^{(k-2)}\right] \mathrm{d}\eta,$$

$$\boldsymbol{\phi}_k = \left\{ \begin{array}{c} \phi_{1k}(x_1,x_2,x_3) \\ \phi_{2k}(x_1,x_2,x_3) \end{array} \right\} = \int_0^{x_3} \left[ \mathbf{L}_2\mathbf{u}^{(k-1)} + \mathbf{S}\boldsymbol{\sigma}_s^{(k-1)} + \mathbf{L}_3\boldsymbol{\sigma}_s^{(k-2)} - \mathbf{D}\phi_{3k} \right] \mathrm{d}\eta.$$

Imposition of the lateral boundary conditions (26) on (38) and (39) leads to the CST type equations with nonhomogeneous terms carried over from the lower-order solution.

$$K_{11}u_1^k + K_{12}u_2^k + K_{13}u_3^k = f_{1k}(x_1,x_2,1) + I_{11}\frac{\partial^2}{\partial\tau_0^2}(u_{3,1}^k) - I_{10}\frac{\partial^2 u_1^k}{\partial\tau_0^2}, \tag{40}$$

$$K_{21}u_1^k + K_{22}u_2^k + K_{23}u_3^k = f_{2k}(x_1,x_2,1) + (I_{11}/r)\frac{\partial^2}{\partial\tau_0^2}(u_{3,2}^k) - I_{10}\frac{\partial^2 u_2^k}{\partial\tau_0^2}, \tag{41}$$

$$\begin{aligned} K_{31}u_1^k + K_{32}u_2^k + \left(K_{33} + K_N\widetilde{T}\right)u_3^k &= f_{3k}(x_1,x_2,1) + \mathbf{D}^{\mathrm{T}}\mathbf{f}_k(x_1,x_2,1) + (s_\alpha/r)f_{1k}(x_1,x_2,1) - I_{20}\frac{\partial^2 u_3^1}{\partial\tau_0^2} \\ &\quad + I_{12}\frac{\partial^2}{\partial\tau_0^2}[\mathbf{D}^{\mathrm{T}}\mathbf{D}u_3^k + (s_\alpha/r)u_{3,1}^k] - I_{11}\frac{\partial^2}{\partial\tau_0^2}[\mathbf{D}^{\mathrm{T}}\mathbf{u}^k + (s_\alpha/r)u_1^k]. \end{aligned} \tag{42}$$

The differential operators $K_{ij}$ for higher-order problems are the same as defined in the leading-order problems. The nonhomogeneous terms at higher-order problems can be calculated from the lower-order solutions. The solution procedure for the leading-order problem can therefore be repeatedly applied for the solution to higher-order problems.

## 5. Thermal dynamic instability analysis

The solution procedure for solving for the thermal dynamic instability of cross-ply laminated conical shells with simply supported boundary conditions is presented as follows.

The elastic moduli for an orthotropic layer are

$$Q_{16} = Q_{26} = Q_{36} = Q_{45} = 0. \tag{43}$$

The simply supported boundary conditions for various orders are specified as:

$$u_2^k = u_3^k = N_1^k = M_1^k = 0 \quad (k = 0,1,2,\ldots), \tag{44}$$

where

$$N_1^k = \int_{-1}^{1} \sigma_1^{(k)}\gamma_\theta\,\mathrm{d}x_3, \qquad M_1^k = \int_{-1}^{1} x_3\sigma_1^{(k)}\gamma_\theta\,\mathrm{d}x_3.$$

According to (4) and (8m), the dimensionless form of external thermal load $\Delta\widetilde{T}$ is expressed in the form of

$$\Delta\widetilde{T} = \alpha_\mathrm{s}\Delta\widetilde{T}_\mathrm{cr}\widetilde{\phi} + \alpha_\mathrm{d}\Delta\widetilde{T}_\mathrm{cr}\widetilde{\phi}\cos(\widetilde{\Omega}\tau_0 - \psi), \tag{45}$$

where $\Delta\widetilde{T}_\mathrm{cr} = \eta\Delta T_\mathrm{cr}/Q\varepsilon^2$; $\widetilde{\Omega} = \Omega R\sqrt{\rho_0/Q}$; the phase angle $\psi$ is a function of $\tau_1, \tau_2, \tau_3, \ldots$, but not of $\tau_0$.

### 5.1. The method of Fourier series expansion

The method of Fourier series expansion is used to eliminate the circumferential coordinate $x_2$ in the formulation. By satisfying the periodicity condition, we let the displacements of the leading order be of the form

$$u_1^0 = \tilde{u}_1^0(x_1, \tau_0, \tau_1, \tau_2, \ldots)\cos nx_2, \tag{46}$$

$$u_2^0 = \tilde{u}_2^0(x_1, \tau_0, \tau_1, \tau_2, \ldots)\sin nx_2, \tag{47}$$

$$u_3^0 = \tilde{u}_3^0(x_1, \tau_0, \tau_1, \tau_2, \ldots)\cos nx_2, \tag{48}$$

where $n$ denotes the circumferential wave number.

Substituting (46)–(48) into (33)–(35) yields the leading-order equations:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 \tilde{u}_1^0}{\partial \tau_0^2} \\ \frac{\partial^2 \tilde{u}_2^0}{\partial \tau_0^2} \\ \frac{\partial^2 \tilde{u}_3^0}{\partial \tau_0^2} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & (k_{33} + k_N \widetilde{T}) \end{bmatrix} \begin{Bmatrix} \tilde{u}_1^0 \\ \tilde{u}_2^0 \\ \tilde{u}_3^0 \end{Bmatrix} = \mathbf{0}, \tag{49}$$

where

$$k_{11} = -\widehat{A}_{11}\partial_{11} + n^2\bar{A}_{66}/r^2 - \widetilde{A}_{11}s_\alpha\partial_1/r + \bar{A}_{22}s_\alpha^2/r^2,$$

$$k_{12} = -n(\widetilde{A}_{12} + \widetilde{A}_{66})\partial_1/r + n(\bar{A}_{22} + \bar{A}_{66})s_\alpha/r^2,$$

$$k_{13} = \widehat{B}_{11}\partial_{111} - n^2\left(\widetilde{B}_{12} + \overline{B}_{66} + \widetilde{B}_{66}\right)\partial_1/r^2 + \widetilde{B}_{11}s_\alpha\partial_{11}/r + n^2\left(\widetilde{B}_{12} + \overline{B}_{22} + \widetilde{B}_{66} + \overline{B}_{66}\right)s_\alpha/r^3 \\ - \left(\widetilde{A}_{12}c_\alpha\sqrt{R/h}/r + \overline{B}_{22}s_\alpha^2/r^2\right)\partial_1 + \bar{A}_{22}c_\alpha s_\alpha\sqrt{R/h}/r^2,$$

$$k_{21} = n\left(\widetilde{A}_{12} + \widetilde{A}_{66}\right)\partial_1/r + n\left(\bar{A}_{22} + \bar{A}_{66}\right)s_\alpha/r^2,$$

$$k_{22} = -\widehat{A}_{66}\partial_{11} + n^2\bar{A}_{22}/r^2 - \widetilde{A}_{66}s_\alpha\partial_1/r + \bar{A}_{66}s_\alpha^2/r^2,$$

$$k_{23} = -n(\widetilde{B}_{12} + \widehat{B}_{66} + \widetilde{B}_{66})\partial_{11}/r + n^3\overline{B}_{22}/r^3 - n(\overline{B}_{22} + \overline{B}_{66} + \widetilde{B}_{66} - 2\widehat{B}_{66})s_\alpha\partial_1/r^2 \\ + n\left[\bar{A}_{22}c_\alpha\sqrt{R/h}/r^2 + (\widetilde{B}_{66} + \overline{B}_{66} - 2\widehat{B}_{66})s_\alpha^2/r^3\right],$$

$$k_{31} = -\widehat{B}_{11}\partial_{111} + n^2\left(\widetilde{B}_{12} + \overline{B}_{66} + \widetilde{B}_{66}\right)\partial_1/r^2 - \left(\widetilde{B}_{11} + \widehat{B}_{11}\right)s_\alpha\partial_{11}/r + n^2\overline{B}_{22}s_\alpha/r^3 \\ + \left(\widetilde{A}_{12}c_\alpha\sqrt{R/h}/r + \overline{B}_{22}s_\alpha^2/r^2\right)\partial_1 + \bar{A}_{22}s_\alpha c_\alpha\sqrt{R/h}/r^2 - \overline{B}_{22}s_\alpha^3/r^3 - \widehat{B}_{11,1}\partial_{11} + \overline{B}_{22,1}s_\alpha^2/r^2 + n^2\overline{B}_{66,1}/r^2,$$

$$k_{32} = -n\left(\widetilde{B}_{12} + \widehat{B}_{66} + \widetilde{B}_{66}\right)\partial_{11}/r + n^3\overline{B}_{22}/r^3 + n\left(\overline{B}_{22} + \overline{B}_{66} - \widetilde{B}_{66}\right)s_\alpha\partial_1/r^2 + n\left(\bar{A}_{22}c_\alpha\sqrt{R/h}/r^2 - \overline{B}_{22}s_\alpha^2/r^3\right) \\ + n\left(\overline{B}_{22,1} + \overline{B}_{66,1}\right)s_\alpha/r^2,$$

$$k_{33} = \widehat{D}_{11}\partial_{1111} - n^2\left(2\widetilde{D}_{12} + \overline{D}_{66} + 2\widetilde{D}_{66} + \widehat{D}_{66}\right)\partial_{11}/r^2 + n^4\overline{D}_{22}/r^4 + \left(\widetilde{D}_{11} + \widehat{D}_{11}\right)s_\alpha\partial_{111}/r \\ + n^2\left(2\widehat{D}_{66} + \widetilde{D}_{66} + \overline{D}_{66} + 2\widetilde{D}_{12}\right)s_\alpha\partial_1/r^3 - \left(2\widetilde{B}_{12}c_\alpha\sqrt{R/h}/r + \overline{D}_{22}s_\alpha^2/r^2\right)\partial_{11} + n^2\Big[2\overline{B}_{22}c_\alpha\sqrt{R/h}/r^3 \\ - \left(2\widetilde{D}_{12} + 2\overline{D}_{22} + \widetilde{D}_{66} + \overline{D}_{66} + 2\widehat{D}_{66}\right)s_\alpha^2/r^4\Big] + \overline{D}_{22}s_\alpha^3\partial_1/r^3 + \bar{A}_{22}\left(c_\alpha\sqrt{R/h}/r\right)^2 - \overline{B}_{22}s_\alpha^2c_\alpha\sqrt{R/h}/r^3 \\ + \widehat{D}_{11,1}\partial_{111} - n^2\overline{D}_{66,1}\partial_1/r^2 + n^2s_\alpha\left(\overline{D}_{22} + \overline{D}_{66}\right)_{,1}/r^3 - s_\alpha^2\overline{D}_{22,1}\partial_1/r^2 + s_\alpha c_\alpha\sqrt{R/h}\overline{B}_{22,1}/r^2,$$

$$k_N = \hat{\gamma}_1\partial_{11} + (\widetilde{\gamma}_1 s_\alpha/r)\partial_1 - (\overline{\gamma}_2/r^2)n^2,$$

$$m_{11} = I_{10}, \quad m_{13} = -I_{11}\partial_1, \quad m_{22} = I_{10}, \quad m_{23} = (I_{11}/r)n, \quad m_{31} = I_{11}[(s_\alpha/r) + \partial_1],$$

$$m_{32} = I_{11}n/r, \quad m_{33} = I_{20} - I_{12}\left[(s_\alpha/r)\partial_1 + \partial_{11} - (n^2/r^2)\right], \quad m_{12} = m_{21} = 0.$$

Eq. (49) are a set of differential equations with variable coefficients. Furthermore, the stiffness coefficients ($\bar{A}_{ij}, \widehat{A}_{ij}, \overline{B}_{ij}, \widehat{B}_{ij}, \overline{D}_{ij}$ and $\widehat{D}_{ij}$) depend on the meridional coordinate $x_1$. The DQ method is adopted for solving the equations.

## 5.2. The method of differential quadrature

The method of DQ (Bellman and Casti, 1971; Bellman et al., 1972) was proposed for the solutions of linear and nonlinear partial differential equations. In the DQ rule, a spatial derivative of an unknown function at a particular sampling point is approximated as a weighted linear sum of the functional values at all the sampling points in the spatial direction. Using the Lagarange polynomials as the test functions, Shu and Richards (1992) presented the expressions of weighting coefficients of first and higher derivatives. The boundary points and the zeros of the Chebyshev functions were suggested to be the sampling points (Bert and Malik, 1997). A comprehensive literature review related to the application of the DQ method in computational mechanics was made by Bert and Malik (1996, 1997). Application of the DQ method to an asymptotic theory for free vibration problems of laminated composite conical shells was made in an earlier paper (Wu and Wu, 2000). Hence, the corresponding expression of the DQ method is not repeated here.

According to the DQ rule, the governing equations and the corresponding boundary conditions can be replaced by a system of simultaneously linear algebraic equations in terms of the mid-surface displacements at all the sampling points. A treatment commonly used in the literature (Du et al., 1994; Shu, 1996) is adopted in the present study.

The first two governing equations in (49) are applied at the interior points ($i = 2, 3, \ldots, N-1$) and the third governing equation is applied at the interior points ($i = 3, 4, \ldots, N-2$). These resulting equations are written by

$$[\mathbf{M_{II}} \quad \mathbf{M_{IB}}]\begin{Bmatrix} \frac{\partial^2 \mathbf{\Delta}_\mathbf{I}^0}{\partial \tau_0^2} \\ \frac{\partial^2 \mathbf{\Delta}_\mathbf{B}^0}{\partial \tau_0^2} \end{Bmatrix} + \left[ \left(\mathbf{K_{II}} + \alpha_s \Delta\widetilde{T}_{cr}\mathbf{K_{NI}}\right) \quad \left(\mathbf{K_{IB}} + \alpha_s\Delta\widetilde{T}_{cr}\mathbf{K_{NB}}\right)\right]\begin{Bmatrix} \mathbf{\Delta}_\mathbf{I}^0 \\ \mathbf{\Delta}_\mathbf{B}^0 \end{Bmatrix} + \left[\alpha_d\Delta\widetilde{T}_{cr}\cos\left(\widetilde{\Omega}\tau_0 - \psi\right)\right]$$
$$\times [\mathbf{K_{NI}} \quad \mathbf{K_{NB}}]\begin{Bmatrix} \mathbf{\Delta}_\mathbf{I}^0 \\ \mathbf{\Delta}_\mathbf{B}^0 \end{Bmatrix} = \mathbf{0}, \tag{50}$$

where $\mathbf{\Delta}_\mathbf{I}^0$ consists of the unknowns $\tilde{u}_1^0$, $\tilde{u}_2^0$, $\tilde{u}_3^0$ at the sampling points $i = 3, 4, \ldots, (N-2)$ and $\tilde{u}_1^0$, $\tilde{u}_2^0$ at the sampling points $i = 2$ and $(N-1)$; $\mathbf{\Delta}_\mathbf{B}^0$ consists of the unknowns $\tilde{u}_1^0$, $\tilde{u}_2^0$, $\tilde{u}_3^0$ at the boundary points and $\tilde{u}_3^0$ at the sampling points $i = 2$ and $(N-1)$.

In accordance with the DQ rule, the boundary conditions (44) at edges ($x_1 = R_1/s_\alpha\sqrt{Rh}$ and $R_2/s_\alpha\sqrt{Rh}$) are rewritten as

$$[\mathbf{K_{BI}} \quad \mathbf{K_{BB}}]\begin{Bmatrix} \mathbf{\Delta}_\mathbf{I}^0 \\ \mathbf{\Delta}_\mathbf{B}^0 \end{Bmatrix} = \mathbf{0}. \tag{51}$$

Eq. (51) can be rewritten as $\mathbf{\Delta}_\mathbf{B}^0 = -\mathbf{K}_\mathbf{BB}^{-1}\mathbf{K}_\mathbf{BI}\mathbf{\Delta}_\mathbf{I}^0$. Substituting the resulting equations into (50) yields

$$\mathbf{A}\frac{\partial^2 \mathbf{\Delta}_\mathbf{I}^0}{\partial \tau_0^2} + (\mathbf{B} + \alpha_\mathrm{s}\Delta\widetilde{T}_\mathrm{cr}\mathbf{C})\mathbf{\Delta}_\mathbf{I}^0 + \left[\alpha_\mathrm{d}\Delta\widetilde{T}_\mathrm{cr}\cos(\widetilde{\Omega}\tau_0 - \psi)\right]\mathbf{C}\mathbf{\Delta}_\mathbf{I}^0 = \mathbf{0}, \tag{52}$$

where $\mathbf{A} = \mathbf{M}_\mathbf{II} - \mathbf{M}_\mathbf{IB}\mathbf{K}_\mathbf{BB}^{-1}\mathbf{K}_\mathbf{BI}$, $\mathbf{B} = \mathbf{K}_\mathbf{II} - \mathbf{K}_\mathbf{IB}\mathbf{K}_\mathbf{BB}^{-1}\mathbf{K}_\mathbf{BI}$ and $\mathbf{C} = \mathbf{K}_\mathbf{NI} - \mathbf{K}_\mathbf{NB}\mathbf{K}_\mathbf{BB}^{-1}\mathbf{K}_\mathbf{BI}$.

### *5.3. Bolotin's method*

Eq. (52) is a system of generalized Mathieu-Hill equations that represents the dynamic instability behavior of laminated conical shells subjected to static and periodic thermal loads.

According to Bolotin's method (Bolotin, 1964), the boundary frequencies of thermal dynamic instability regions can be determined by letting $\mathbf{\Delta}_\mathbf{I}^0$ as the following form

$$\mathbf{\Delta}_\mathbf{I}^0 = \sum_{k=1,2,\ldots}^{\infty}\left[\mathbf{a}_{2k-1}^0\sin\frac{(2k-1)(\widetilde{\Omega}\tau_0 - \psi)}{2} + \mathbf{b}_{2k-1}^0\cos\frac{(2k-1)(\widetilde{\Omega}\tau_0 - \psi)}{2}\right], \tag{53}$$

$$\mathbf{\Delta}_\mathbf{I}^0 = \mathbf{b}_0^0 + \sum_{k=1,2,\ldots}^{\infty}\left[\mathbf{a}_{2k}^0\sin k(\widetilde{\Omega}\tau_0 - \psi) + \mathbf{b}_{2k}^0\cos k(\widetilde{\Omega}\tau_0 - \psi)\right]. \tag{54}$$

Eqs. (53) and (54) represent the infinite terms of periodic functions of time with period $4\pi/\widetilde{\Omega}$ and $2\pi/\widetilde{\Omega}$, respectively. It is well known that the solutions with period $4\pi/\widetilde{\Omega}$ are of great practical importance due to the fact that the unstable regions obtained using (53) are usually much larger than those regions obtained using (54). Hence, the former is denoted as the primary instability region and the latter is the secondary instability region. In view of the rapid convergence of Bolotin's method, only the first few terms of (53) and (54) will be adopted in the present study. The convergence of the $K$-term approximate solutions will be examined.

#### *5.3.1. The primary instability regions*

Substituting (53) in (52), simplifying, and grouping the sine and cosine terms lead to two sets of linear algebraic equations in $\mathbf{a}_{2k-1}^0$ and $\mathbf{b}_{2k-1}^0 (k = 1, 2, \ldots, K)$ for each $K$-term solution. The resulting equations are given by

For the one-term solution ($K = 1$):

$$\sin\frac{(\widetilde{\Omega}\tau_0 - \psi)}{2}\text{ term:}\quad \left[\mathbf{B} + \left(\alpha_\mathrm{s} - \frac{1}{2}\alpha_\mathrm{d}\right)\Delta\widetilde{T}_\mathrm{cr}\mathbf{C} - \frac{\widetilde{\Omega}^2}{4}\mathbf{A}\right]\mathbf{a}_1^0 = \mathbf{0}; \tag{55a}$$

$$\cos\frac{(\widetilde{\Omega}\tau_0 - \psi)}{2}\text{ term:}\quad \left[\mathbf{B} + \left(\alpha_\mathrm{s} + \frac{1}{2}\alpha_\mathrm{d}\right)\Delta\widetilde{T}_\mathrm{cr}\mathbf{C} - \frac{\widetilde{\Omega}^2}{4}\mathbf{A}\right]\mathbf{b}_1^0 = \mathbf{0}. \tag{55b}$$

For the two-term solution ($K = 2$):

$$\sin\frac{(\widetilde{\Omega}\tau_0 - \psi)}{2}\text{ and }\sin\frac{3(\widetilde{\Omega}\tau_0 - \psi)}{2}\text{ terms:}\quad \begin{bmatrix} \mathbf{B} + \left(\alpha_\mathrm{s} - \frac{1}{2}\alpha_\mathrm{d}\right)\Delta\widetilde{T}_\mathrm{cr}\mathbf{C} - \frac{\widetilde{\Omega}^2}{4}\mathbf{A} & \frac{\alpha_\mathrm{d}\Delta\widetilde{T}_\mathrm{cr}}{2}\mathbf{C} \\ \frac{\alpha_\mathrm{d}\Delta\widetilde{T}_\mathrm{cr}}{2}\mathbf{C} & \mathbf{B} + \alpha_\mathrm{s}\Delta\widetilde{T}_\mathrm{cr}\mathbf{C} - \frac{9\widetilde{\Omega}^2}{4}\mathbf{A} \end{bmatrix} \times \begin{Bmatrix} \mathbf{a}_1^0 \\ \mathbf{a}_3^0 \end{Bmatrix} = \mathbf{0}, \tag{56a}$$

$$\cos\frac{(\widetilde{\Omega}\tau_0-\psi)}{2}\ \text{and}\ \cos\frac{3(\widetilde{\Omega}\tau_0-\psi)}{2}\ \text{terms:}\quad \begin{bmatrix} \mathbf{B}+\left(\alpha_s+\frac{1}{2}\alpha_d\right)\Delta\widetilde{T}_{cr}\mathbf{C}-\frac{\widetilde{\Omega}^2}{4}\mathbf{A} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} \\ \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C}-\frac{9\widetilde{\Omega}^2}{4}\mathbf{A} \end{bmatrix} \times \begin{Bmatrix} \mathbf{b}_1^0 \\ \mathbf{b}_3^0 \end{Bmatrix} = \mathbf{0}. \tag{56b}$$

For the $K$-term solution:

$$\sin\frac{\left(\widetilde{\Omega}\tau_0-\psi\right)}{2},\quad \sin\frac{3\left(\widetilde{\Omega}\tau_0-\psi\right)}{2},\ldots\ \text{and}\ \sin\frac{(2k-1)\left(\widetilde{\Omega}\tau_0-\psi\right)}{2}\ \text{terms:}$$

$$\begin{bmatrix} \mathbf{B}+\left(\alpha_s-\frac{1}{2}\alpha_d\right)\Delta\widetilde{T}_{cr}\mathbf{C}-\frac{\widetilde{\Omega}^2}{4}\mathbf{A} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{0} & \cdots & \mathbf{0} \\ \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C}-\frac{9\widetilde{\Omega}^2}{4}\mathbf{A} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \ddots & \vdots \\ \mathbf{0} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C}-\frac{25\widetilde{\Omega}^2}{4}\mathbf{A} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C}-\frac{(2k-1)^2\widetilde{\Omega}^2}{4}\mathbf{A} \end{bmatrix} \times \begin{Bmatrix} \mathbf{a}_1^0 \\ \mathbf{a}_3^0 \\ \mathbf{a}_5^0 \\ \vdots \\ \mathbf{a}_{2k-1}^0 \end{Bmatrix} = \mathbf{0}, \tag{57a}$$

$$\cos\frac{\left(\widetilde{\Omega}\tau_0-\psi\right)}{2},\quad \cos\frac{3\left(\widetilde{\Omega}\tau_0-\psi\right)}{2},\ldots\ \text{and}\ \cos\frac{(2k-1)\left(\widetilde{\Omega}\tau_0-\psi\right)}{2}\ \text{terms:}$$

$$\begin{bmatrix} \mathbf{B}+\left(\alpha_s+\frac{1}{2}\alpha_d\right)\Delta\widetilde{T}_{cr}\mathbf{C}-\frac{\widetilde{\Omega}^2}{4}\mathbf{A} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{0} & \cdots & \mathbf{0} \\ \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C}-\frac{9\widetilde{\Omega}^2}{4}\mathbf{A} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \ddots & \vdots \\ \mathbf{0} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C}-\frac{25\widetilde{\Omega}^2}{4}\mathbf{A} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C}-\frac{(2k-1)^2\widetilde{\Omega}^2}{4}\mathbf{A} \end{bmatrix} \times \begin{Bmatrix} \mathbf{b}_1^0 \\ \mathbf{b}_3^0 \\ \mathbf{b}_5^0 \\ \vdots \\ \mathbf{b}_{2k-1}^0 \end{Bmatrix} = \mathbf{0}. \tag{57b}$$

Observation of (55a)–(57b) reveals that the coefficient matrices related to sine and cosine terms appear in a recurrent pattern through the $K$-term solution. For a fixed value of circumferential wave number $n$, we can determine one-term approximate solutions for upper and lower bounds of the instability region by setting the determinants of coefficients of (55a) and (55b) equal to zero, respectively. The solutions can then be successively modified by using (56a)–(57a) and (56b)–(57b). The first instability region corresponding to the smallest eigenvalues is of great practical importance so that it is also denoted as the principal instability region (Bolotin, 1964) and is the main concerns in the literature.

*5.3.2. The secondary instability regions*

Substituting (54) in (52), simplifying, and grouping the sine and cosine terms lead to two sets of linear algebraic equations in $\mathbf{a}_{2k}^0$ $(k = 1, 2, \dots, K)$ and $\mathbf{b}_{2k}^0$ $(k = 0, 1, 2, \dots, K)$ for each $K$-term solution. The resulting equations are given by

For the one-term solution ($K = 1$):

$$\sin\left(\widetilde{\Omega}\tau_0 - \psi\right)\text{term:}\quad \left[\mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} - \widetilde{\Omega}^2\mathbf{A}\right]\mathbf{a}_2^0 = \mathbf{0}, \tag{58a}$$

$$1 \text{ and } \cos\left(\widetilde{\Omega}\tau_0 - \psi\right)\text{terms:}\quad \begin{bmatrix} \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} & \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} \\ \alpha_d \Delta \widetilde{T}_{cr}\mathbf{C} & \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} - \widetilde{\Omega}^2\mathbf{A} \end{bmatrix} \begin{Bmatrix} \mathbf{b}_0^0 \\ \mathbf{b}_2^0 \end{Bmatrix} = \mathbf{0}. \tag{58b}$$

For the two-term solution ($K = 2$):

$$\sin\left(\widetilde{\Omega}\tau_0 - \psi\right) \text{ and } \sin 2\left(\widetilde{\Omega}\tau_0 - \psi\right)\text{terms:}\quad \begin{bmatrix} \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} - \widetilde{\Omega}^2\mathbf{A} & \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} \\ \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} - 4\widetilde{\Omega}^2\mathbf{A} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_2^0 \\ \mathbf{a}_4^0 \end{Bmatrix} = \mathbf{0}, \tag{59a}$$

$$1, \cos\left(\widetilde{\Omega}\tau_0 - \psi\right) \text{ and } \cos 2\left(\widetilde{\Omega}\tau_0 - \psi\right)\text{terms:}$$
$$\begin{bmatrix} \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} & \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{0} \\ \alpha_d \Delta \widetilde{T}_{cr}\mathbf{C} & \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} - \widetilde{\Omega}^2\mathbf{A} & \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} \\ \mathbf{0} & \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} - 4\widetilde{\Omega}^2\mathbf{A} \end{bmatrix} \begin{Bmatrix} \mathbf{b}_0^0 \\ \mathbf{b}_2^0 \\ \mathbf{b}_4^0 \end{Bmatrix} = \mathbf{0}. \tag{59b}$$

For the $K$-term solution:

$$\sin\left(\widetilde{\Omega}\tau_0 - \psi\right), \sin 2\left(\widetilde{\Omega}\tau_0 - \psi\right), \dots \text{ and } \sin 2k\left(\widetilde{\Omega}\tau_0 - \psi\right)\text{terms:}$$
$$\begin{bmatrix} \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} - \widetilde{\Omega}^2\mathbf{A} & \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{0} & \cdots & \mathbf{0} \\ \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} - 4\widetilde{\Omega}^2\mathbf{A} & \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} & \ddots & \vdots \\ \mathbf{0} & \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} - 9\widetilde{\Omega}^2\mathbf{A} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{\alpha_d \Delta \widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr}\mathbf{C} - k^2\widetilde{\Omega}^2\mathbf{A} \end{bmatrix}$$
$$\times \begin{Bmatrix} \mathbf{a}_2^0 \\ \mathbf{a}_4^0 \\ \mathbf{a}_6^0 \\ \vdots \\ \mathbf{a}_{2k}^0 \end{Bmatrix} = \mathbf{0}, \tag{60a}$$

$1, \cos\left(\widetilde{\Omega}\tau_0-\psi\right), \cos 2\left(\widetilde{\Omega}\tau_0-\psi\right), \ldots$ and $\cos 2k\left(\widetilde{\Omega}\tau_0-\psi\right)$ terms:

$$\begin{bmatrix} \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{0} & \cdots & \mathbf{0} \\ \alpha_d\Delta\widetilde{T}_{cr}\mathbf{C} & \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C}-\widetilde{\Omega}^2\mathbf{A} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \ddots & \vdots \\ \mathbf{0} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C}-4\widetilde{\Omega}^2\mathbf{A} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{\alpha_d\Delta\widetilde{T}_{cr}}{2}\mathbf{C} & \mathbf{B}+\alpha_s\Delta\widetilde{T}_{cr}\mathbf{C}-k^2\widetilde{\Omega}^2\mathbf{A} \end{bmatrix}$$

$$\times \begin{Bmatrix} \mathbf{b}_0^0 \\ \mathbf{b}_2^0 \\ \mathbf{b}_4^0 \\ \vdots \\ \mathbf{b}_{2k}^0 \end{Bmatrix} = \mathbf{0}. \tag{60b}$$

Again, observation of (58a)–(60b) reveals that the coefficient matrices related to sine and cosine terms appear in a recurrent pattern through the $K$-term solution. By setting the determinants of the coefficients of (58a) and (58b) equal to zero, respectively, we can determine one-term approximate solutions for upper and lower bounds of the secondary instability region for a fixed value of $n$. The solutions can then be successively modified by using (59a)–(60a) and (59b)–(60b).

## 5.4. The orthonormality and solvability conditions

Since a set of unique solution is required in the present analysis, the modal unknowns are normalized by imposing the orthonormality conditions:

$$\left(\mathbf{\Delta}_\mathbf{I}^0+\varepsilon^2\mathbf{\Delta}_\mathbf{I}^1+\varepsilon^4\mathbf{\Delta}_\mathbf{I}^2+\cdots\right)^\mathrm{T}\cdot\left(\mathbf{\Delta}_\mathbf{I}^0+\varepsilon^2\mathbf{\Delta}_\mathbf{I}^1+\varepsilon^4\mathbf{\Delta}_\mathbf{I}^2+\cdots\right)=1. \tag{61}$$

According to (61), the orthonormality conditions for various orders are specified as

$$\varepsilon^0\text{-order:}\quad (\mathbf{\Delta}_\mathbf{I}^0)^\mathrm{T}\cdot\mathbf{\Delta}_\mathbf{I}^0=1; \tag{62}$$

$$\varepsilon^2\text{-order:}\quad \begin{aligned}&(\mathbf{\Delta}_\mathbf{I}^0)^\mathrm{T}\cdot\mathbf{\Delta}_\mathbf{I}^0=1,\\&(\mathbf{\Delta}_\mathbf{I}^0)^\mathrm{T}\cdot\mathbf{\Delta}_\mathbf{I}^1=0;\end{aligned} \tag{63}$$

$$\varepsilon^4\text{-order:}\quad \begin{aligned}&(\mathbf{\Delta}_\mathbf{I}^0)^\mathrm{T}\cdot\mathbf{\Delta}_\mathbf{I}^0=1,\\&(\mathbf{\Delta}_\mathbf{I}^0)^\mathrm{T}\cdot\mathbf{\Delta}_\mathbf{I}^1=0,\\&2(\mathbf{\Delta}_\mathbf{I}^0)^\mathrm{T}\cdot\mathbf{\Delta}_\mathbf{I}^2+(\mathbf{\Delta}_\mathbf{I}^1)^\mathrm{T}\cdot\mathbf{\Delta}_\mathbf{I}^1=0;\ldots\text{etc.}\end{aligned} \tag{64}$$

Carrying on the solution to order $\varepsilon^2$, we let the displacements of $\varepsilon^2$ order be as follows.

$$u_1^1(x_1,x_2,\tau_0,\tau_1,\ldots)=\tilde{u}_1^1(x_1,\tau_0,\tau_1,\ldots)\cos nx_2, \tag{65}$$

$$u_2^1(x_1,x_2,\tau_0,\tau_1,\ldots)=\tilde{u}_2^1(x_1,\tau_0,\tau_1,\ldots)\sin nx_2, \tag{66}$$

$$u_3^1(x_1, x_2, \tau_0, \tau_1, \dots) = \tilde{u}_3^1(x_1, \tau_0, \tau_1, \dots) \cos nx_2. \tag{67}$$

Substituting (65)–(67) in the governing equations of $\varepsilon^2$-order ($k = 1$ in (40)–(42)) leads to

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 \tilde{u}_1^1}{\partial \tau_0^2} \\ \frac{\partial^2 \tilde{u}_2^1}{\partial \tau_0^2} \\ \frac{\partial^2 \tilde{u}_3^1}{\partial \tau_0^2} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & \left(k_{33} + k_N \widetilde{T}\right) \end{bmatrix} \begin{Bmatrix} \tilde{u}_1^1 \\ \tilde{u}_2^1 \\ \tilde{u}_3^1 \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{11}(x_3 = 1) \frac{\partial \psi_i}{\partial \tau_1} + \tilde{f}_{11}(x_3 = 1) \\ \hat{f}_{21}(x_3 = 1) \frac{\partial \psi_i}{\partial \tau_1} + \tilde{f}_{21}(x_3 = 1) \\ \hat{h}_{11}(x_3 = 1) \frac{\partial \psi_i}{\partial \tau_1} + \tilde{h}_{11}(x_3 = 1) \end{Bmatrix}, \tag{68}$$

where $\hat{h}_{31} = \hat{f}_{31} + \hat{f}_{11,1} + n\hat{f}_{21}/r + s_\alpha \hat{f}_{11}/r$ and $\tilde{h}_{31} = \tilde{f}_{31} + \tilde{f}_{11,1} + n\tilde{f}_{21}/r + s_\alpha \tilde{f}_{11}/r$.

After applying the DQ rule to (68) and following the similar procedure as was done through (50)–(52), we obtain

$$\mathbf{A} \frac{\partial^2 \mathbf{\Delta}_\mathbf{I}^1}{\partial \tau_0^2} + \left(\mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr} \mathbf{C}\right) \mathbf{\Delta}_\mathbf{I}^1 + \left[\alpha_d \Delta \widetilde{T}_{cr} \cos\left(\widetilde{\Omega}_i \tau_0 - \psi_i\right)\right] \mathbf{C} \mathbf{\Delta}_\mathbf{I}^1 = \hat{\mathbf{f}} \left(\frac{\partial \psi_i}{\partial \tau_1}\right) + \tilde{\mathbf{f}}. \tag{69}$$

Eq. (69) is solvable if and only if the solvability condition is satisfied (Nayfeh, 1993). The solvability condition is given by

$$\mathbf{\Lambda}_i^\mathrm{T} \cdot \left[\hat{\mathbf{f}} \left(\frac{\partial \psi_i}{\partial \tau_1}\right) + \tilde{\mathbf{f}}\right] = 0, \tag{70}$$

where $\mathbf{\Lambda}_i$ denotes the eigenvectors corresponding to the eigenvalues $\widetilde{\Omega}_i$.

The dependence of $\psi_i$ upon $\tau_1$ can then be determined as

$$\psi_i = -\lambda_i \tau_1 + \tilde{\psi}_i(\tau_2, \tau_3, \dots), \tag{71}$$

where $\lambda_i = (\mathbf{\Lambda}_i^\mathrm{T} \cdot \tilde{\mathbf{f}})/(\mathbf{\Lambda}_i^\mathrm{T} \cdot \hat{\mathbf{f}})$; $\tilde{\psi}_i$ are functions of the scales $\tau_2$, $\tau_3, \dots$, that can be determined at the next-order level.

With (71) and the relation $\tau_1 = \varepsilon^2 \tau_0 = (h/R)\tau_0$, the time functions of all field variables are now expressed in terms of $\sin((2k-1)[(\widetilde{\Omega} + \lambda h/R)\tau_0 - \tilde{\psi}]/2)$, $\cos((2k-1)[(\widetilde{\Omega} + \lambda h/R)\tau_0 - \tilde{\psi}]/2)$, $\sin k[(\widetilde{\Omega} + \lambda h/R)\tau_0 - \tilde{\psi}]$ and $\cos k[(\widetilde{\Omega} + \lambda h/R)\tau_0 - \tilde{\psi}]$ . Hence, the upper and lower bounds of the dynamic instability regions at the $\varepsilon^2$-order level have been modified as

$$\widetilde{\Omega}_i + \lambda_i(h/R). \tag{72}$$

Substituting (71) in (69) yields

$$\mathbf{A} \frac{\partial^2 \mathbf{\Delta}_\mathbf{I}^1}{\partial \tau_0^2} + \left(\mathbf{B} + \alpha_s \Delta \widetilde{T}_{cr} \mathbf{C}\right) \mathbf{\Delta}_\mathbf{I}^1 + \left[\alpha_d \Delta \widetilde{T}_{cr} \cos\left(\widetilde{\Omega}_i \tau_0 - \psi_i\right)\right] \mathbf{C} \mathbf{\Delta}_\mathbf{I}^1 = -\lambda_i \hat{\mathbf{f}} + \tilde{\mathbf{f}}. \tag{73}$$

By solving (73) with the orthonormality conditions (63), we can uniquely determine the values of $\mathbf{\Delta}_\mathbf{I}^1$ which denote the first-order modifications to the modal unknowns $\mathbf{\Delta}_\mathbf{I}^0$.

In view of the recurrence relationship between the leading-order and the higher-order problems, the solution procedure can be continued to higher-order levels in a similar way.

## 6. Illustrative example

The thermally induced dynamic instability problems of simply supported, cross-ply laminated conical shells under static and periodic thermal loads are considered in Table 1 and Figs. 2 and 3. In the computations the material properties are given by $E_L/E_T = 25$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.2$, $\nu_{LT} = \nu_{TT} = 0.25$, $\alpha_T = 3\alpha_L$, $E_T = 6.89 \times 10^6$ kN/m$^2$ (or $10^6$ psi), $\alpha_L = 6.3 \times 10^{-6}$ 1/°C, and $\rho/\rho_0 = 1$. The frequency parameter $\overline{\Omega}$ is normalized as $\overline{\Omega} = \Omega R_2\sqrt{2\rho h/A_{11}}$.

Before we proceed to the thermal dynamic instability analysis, a counterpart problem has to be analyzed in advance where the identical shell is subjected to a static temperature field

$$\frac{(T_o + T_i)}{2} + \frac{(T_o - T_i)}{2h}\zeta \quad \text{and} \quad T_i = 3T_o \tag{74}$$

in which $T_o$ and $T_i$ denote the temperature change at the outer and inner surfaces of the shell, respectively. Through nondimensionalization and normalization, we have $\tilde{\phi}(x_3) = \sqrt{3/26}(2 - x_3)$ and $\Delta T_{cr} = \sqrt{26/3}(T_o)_{cr}$ at the critical state. According to the results of a static thermoelastic buckling analysis (Wu and Chiu, 2001), the temperature fields at the critical state are [166.7 °C $\tilde{\phi}(x_3)$] (i.e., $\Delta T_{cr} = 166.7$ °C) for [0/90] and [265.8 °C $\tilde{\phi}(x_3)$] (i.e., $\Delta T_{cr} = 265.8$ °C) for [0/90/0/90] laminated shells. The circumferential wave numbers for the corresponding buckling modes in both cases are computed to be three (i.e., $n = 3$).

In the present analysis, $\alpha_s$ and $\alpha_d$ portions of the temperature field at the critical state in the thermoelastic buckling analysis are taken as the magnitudes of static and periodic thermal loads, respectively, where $\alpha_s + \alpha_d \leqslant 1$. Hence, the temperature field is given as

Table 1
The first primary and secondary instability regions for the cross-ply conical shells

| Laminates | Instability regions | Trigonometric functions | K-term approximations | Present asymptotic solutions | | |
|---|---|---|---|---|---|---|
| | | | | $\varepsilon^0$ | $\varepsilon^2$ | $\varepsilon^4$ |
| [0/90] | Primary | Sin | 1 | 0.3663 | 0.3555 | 0.3556 |
| | | | 2 | 0.3682 | 0.3626 | 0.3626 |
| | | | 3 | 0.3682 | 0.3635 | 0.3633 |
| | | Cos | 1 | 0.2391 | 0.2287 | 0.2290 |
| | | | 2 | 0.2467 | 0.2399 | 0.2398 |
| | | | 3 | 0.2467 | 0.2410 | 0.2406 |
| | Secondary | Sin | 1 | 0.1548 | 0.1496 | 0.1497 |
| | | | 2 | 0.1592 | 0.1548 | 0.1548 |
| | | | 3 | 0.1594 | 0.1552 | 0.1551 |
| | | Cos | 1 | 0.1269 | 0.1219 | 0.1220 |
| | | | 2 | 0.1333 | 0.1284 | 0.1284 |
| | | | 3 | 0.1336 | 0.1290 | 0.1287 |
| [0/90/0/90] | Primary | Sin | 1 | 0.4431 | 0.4244 | 0.4246 |
| | | | 2 | 0.4457 | 0.4395 | 0.4395 |
| | | | 3 | 0.4457 | 0.4402 | 0.4400 |
| | | Cos | 1 | 0.2656 | 0.2447 | 0.2450 |
| | | | 2 | 0.2789 | 0.2682 | 0.2678 |
| | | | 3 | 0.2790 | 0.2693 | 0.2681 |
| | Secondary | Sin | 1 | 0.1827 | 0.1732 | 0.1734 |
| | | | 2 | 0.1890 | 0.1837 | 0.1837 |
| | | | 3 | 0.1891 | 0.1840 | 0.1837 |
| | | Cos | 1 | 0.1360 | 0.1258 | 0.1261 |
| | | | 2 | 0.1506 | 0.1429 | 0.1427 |
| | | | 3 | 0.1510 | 0.1433 | 0.1426 |

$2h/R_1 = 0.1$, $L/R_1 = 5$, $N = 21$, $n = 3$, $\alpha = 45°$, $\alpha_s = 0.2$, $\alpha_d = 0.8$; $\overline{\Omega} = \Omega R_2\sqrt{2\rho h/A_{11}}$.

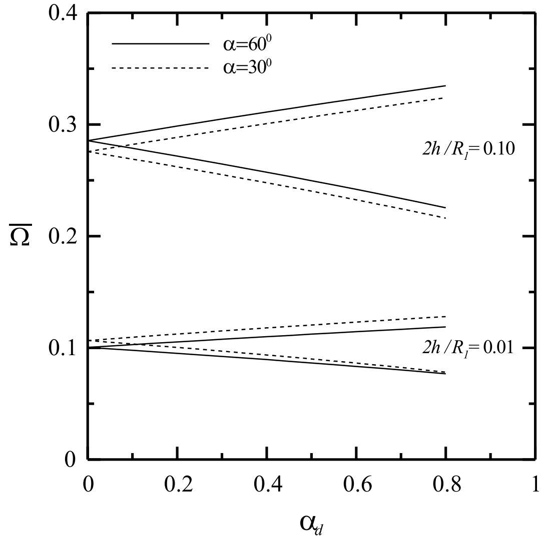


Fig. 2. The principal instability regions of [0/90/0] laminated conical shells.

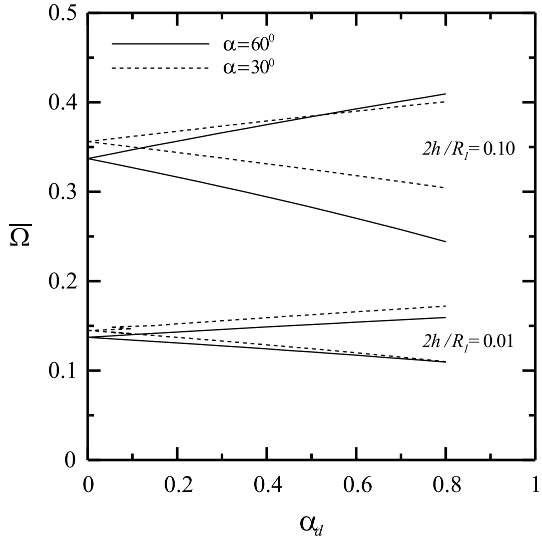


Fig. 3. The principal instability regions of [0/90/0/90] laminated conical shells.

$$\Delta T = \alpha_s \Delta T_{cr} \left\lfloor \sqrt{3/26}(2 - x_3) \right\rfloor + \alpha_d \Delta T_{cr} \left\lfloor \sqrt{3/26}(2 - x_3) \right\rfloor \cos \Omega t. \tag{75}$$

The boundary frequencies of first primary and secondary instability regions for [0/90] and [0/90/0/90] laminated conical shells are given in Table 1. The geometry parameters are $\alpha = 45°$, $L/R_1 = 5$, $2h/R_1 = 0.1$. The static and dynamic parameters of the temperature field are taken as $\alpha_s = 0.2$ and $\alpha_d = 0.8$, respectively. The convergence of the present asymptotic theory and Bolotin's method is examined. It is shown that the convergent solution is obtained at the $\varepsilon^4$-order level with the three-term approximations. It is well known that for $[0/90]_m$ antisymmetric laminates of constant thickness, the extension-bending coupling stiffnesses will decrease as the number of layers (i.e., $2m$) increase. The effect of extension-bending coupling stiffnesses on the boundary frequencies of instability regions can then be evaluated by observing the results of [0/90] and [0/90/0/90] laminated shells. It is noted that the boundary frequencies of the instability regions decrease as the extension-bending coupling stiffnesses increase.

A parametric study for thermal dynamic instability of [0/90/0] and [0/90/0/90] laminated shells is shown in Figs. 2 and 3, respectively. The variations with semivertex angle $\alpha$, thickness-to-radius ratio $2h/R_1$ and dynamic parameter $\alpha_d$ on the principal instability regions are shown. The geometry parameters are $\alpha = 30°$, $60°$; $L/R_1 = 5$; $2h/R_1 = 0.01$, $0.1$. The static and dynamic parameters of the thermal field are taken as $\alpha_s = 0.2$ and $\alpha_d = 0.2$–$0.8$. It is noted that the boundary frequencies of the dynamic instability regions increase as the thickness-to-radius ratio increase. The effect of transverse deformation on the boundary frequencies is much notable than the change of the semivertex angle. The range of the dynamic instability regions increase as the dynamic parameter of the temperature field increase.

## 7. Conclusions

In conjunction with the method of DQ and Bolotin's method, the asymptotic solution for thermally induced dynamic instability of laminated composite conical shells under static and periodic thermal loads is presented. The temperature field is considered a periodic function in time and a certain distributed function in the thickness direction. The convergent solution of the present asymptotic theory is obtained at the $\varepsilon^4$-order level with the three-term approximations. Furthermore, the present asymptotic formulation is applicable to the free vibration analysis of the laminated conical shells in the absence of thermal loads.

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